Symmetry of the Fréedericksz transition in nonchiral nematic liquid crystals

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The Fréedericksz transition in a nonuniaxial, nonchiral nematic liquid crystal (LC) in the presence of the electric **E** field is considered using the Curie symmetry principle. All possible symmetries of (i) the LC point symmetry group G_N, (ii) the strong boundary LC orientation at the infinite plane-parallel plates of slab, and (iii) the direction of the **E** field are analyzed. The free energy polynomial $J(c_i)$ is expressed in terms of the invariant polynomials of components of the three-dimensional axial vector **c**. Possible primary and secondary bifurcations are determined for all classes of nonchiral nematic LC (biaxial, tetrahedral, cubic, and icosahedral). It is shown that different kinds of such LCs, subjected to the **E** fields of different orientations, can be described by the same polynomial $J(c_i)$, invariant with respect to the action of a symmetry group $G_{\rm Fr}$ of the Fréedericksz transition. In the framework of the symmetry approach, the influence of the thermal fluctuations of the nematic directors on the Fréedericksz transition is studied and mean squares of these fluctuations are found.

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I. INTRODUCTION

The absence of translational symmetry in the nematic phase of a liquid crystal (LC) admits the point symmetry groups—subgroups of O(3)—including the groups forbidden in crystalline lattice. The existence of nonuniaxial nematic LCs was predicted [1,2] long before their discovery in the lyotropic mesophase [3]. Later, using the traditional conoscopic and calorimetric measurments, as well as NMR and x-ray diffraction, various nonuniaxial phases in thermotropic nematic LCs were identified: monoclinic [4,5], rhombic [6,7], tetragonal [8], and cubic [9]. The properties of the hypothetical icosahedral nematic LC were described [10] and its orientational order parameter was constructed [11]. Recently, the tetrahedral nematic LC (not yet observed) was discussed in detail [12]. The experimental data on nonuniaxial nematic LCs were first reviewed in [13]. The rhombohedral, tetragonal, and cubic phases were also found in lyotropic LCs [14]. This variety of nematic LC of different point symmetry groups motivated the development of a theory of the physical properties (elasticity, flexoelectricity, and hydrodynamics) for nematic LCs with arbitrary point symmetry group [15–17]. The theory of linear defects for most of the point groups was developed [18] in the framework of a homotopical approach.

A great variety of nonuniaxial, nematic LC symmetry groups may cause difficulties in their experimental identification in the vicinity of a phase transition from isotropic liquid. Therefore it is important to describe the behavior of nonuniaxial nematic LCs in an external electric field **E**. The effect of the electric field has not been largely studied experimentally in nonuniaxial LCs. Usually an applied electric field was used as an orientation tool in biaxial nematic LCs [5,19] in order to provide the conoscopic observation. In the case of a chiral, nonuniaxial LCs, cubic blue phases exhibit different phase transitions under an electric field into uniaxial blue phases, hexagonal [20] or tetragonal [21]. Nevertheless, our knowledge about the main physical properties, e.g., elasticity, of the nonuniaxial nematic LCs is far from complete, which we have for the uniaxial nematics. From the experimental point of view, the study of the orientational instability of a nonuniaxial nematic LC under an electric field (known as the *Fréedericksz structural transition*) provides a sensitive tool for finding some features of the LC inner symmetry as well as the elasticity moduli of LCs.

In the past decade, the Fréedericksz transition in a nonuniaxial nematic LC has been considered first for the cubic [22] and hexagonal [23] nematic phases. Later the respective experiments were carried out in blue cubic phases of LCs [24]: the orientation of cubic LCs predicted in [22] is precisely the same as that observed in [24]. The theoretical approach [22,23] was restricted to finding the critical values E_1 of the fields. The stability of inhomogeneous nematic structures appearing in fields greater than E_1 was not considered. In terms of bifurcation theory, the study was restricted to finding only the primary bifurcation points. The fields beyond the primary bifurcation have been analyzed for rhombic [25] and tetrahedral [12] nematic LCs subjected to the external field **E**. The existence of two successive structural transitions (the primary E_1 and secondary E_2 bifurcation points) was shown and the necessary conditions for the existence of a nonzero threshold transition were found. It was shown that the field **E** should be directed along the rotational axes of an unperturbed nematic LC. The generalization of this problem to the nonuniaxial nematic LC of arbitrary point symmetry group G_N is part of the present author's program of constructing the continuous theory of nematic LCs of arbitrary symmetry [11,12,17,25]. In the present paper, arbitrary homogeneous boundary conditions and the arbitrary direction of the **E** field are taken into account in addition to the

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arbitrary point group G_N. This variety of initial conditions demands a more general approach in the analysis than the cumbersome variational methods used in [12,22,23,25]. The approach taken is based on the Curie symmetry principle and the theory of an integrity basis of invariant polynomials [26].

The influence of thermal fluctuations of the nonuniaxial nematic directors on the field dependence of the order parameter near threshold is of special interest. In the mid 1970s Guyon [27] showed that the Fréedericksz transition in the uniaxial nematic LC cell is analogous to the second-order phase transition in thermodynamics. This approach was extended further in [28] to the analogs of the first-order transition and the isostructural transition. The next natural step was to consider thermal fluctuations of the uniaxial nematic director: it was shown [29] that the Fréedericksz transition in the ordinary uniaxial nematic ($G_N = D_{\infty h}$) with homeotropical boundary orientation is equivalent to the two-dimensional continuous Ising model ($d = 2, n = 1$) with one-component spins, which leads to the corresponding critical Onsager indices in the vicinity of the threshold field $E_{\rm th}$. The experimental data [30] on the anomalous birefringence in uniaxial nematic LC in the vicinity of the Fréedericksz transition also evidenced that consideration of only a uniform static tilt of the director might be insufficient for a correct explanation of the experimental results. Recently [31] the energy of director fluctuations at the Fréedericksz transition in uniaxial nematic LC was estimated. The influence of the symmetry of nonuniaxial nematic LCs on the choice of the model of a phase transition equivalent to the Fréedericksz transition was not discussed. [An equivalence of the Fréedericksz transition in tetragonal nematic LCs ($G_N = D_{4h}$) to the two-dimensional continuous XY model with two-component spins ($d = 2, n = 2$) was suggested [32].]

We restrict our treatment to study of the Fréedericksz transitions in nonchiral nematic LCs with tensor order parameters Q_w of w rank, where $w \geq 2$, since the chirality of the nematic LC gives rise to the spatioperiodical distortion of Helfrich-Hurault type [33,34]. Thus we do not deal with LCs of pyroelectric symmetry classes. The nonchiral LC symmetry groups are [12] *uniaxial* ($D_{kd}, D_{(k+1)h}, k \geq 2$), *biaxial* ($D_{2h}$), *tetrahedral* ($T_d$), *cubic* ($T_h, O_h$), and *icosahedral* ($Y_h$).

The objective of the present work is to describe possible nonchiral nematic structures and types of the Fréedericksz transitions between them in the presence of the electric field. All possible symmetries of the nonchiral LC point group G_N and of the homogeneous directions of $\mathbf{E}$ field are considered. The strong anchoring of the nematic directors at the boundaries and their arbitrary homogeneous orientations are assumed. The influence of the thermal fluctuations of the nematic directors on the Fréedericksz transition is taken into account.

II. STATEMENT OF THE PROBLEM

In the absence of an external field $\mathbf{E}$, the distribution of LC directors is homogeneous in the infinite plane-parallel layer of a nonchiral nematic LC belonging to one of the symmetry groups G_N listed above. Now suppose that the layer is subjected to a homogeneous external electric field of arbitrary orientation. The strong anchoring of the directors at the boundaries is assumed to be of an arbitrary homogeneous orientation. In a general, nonsymmetrical case, this leads to a wide variety of orientation parameters and, as a consequence, to the continuous, nonthreshold structural Fréedericksz transition. For example, as shown in [25], the nonzero field threshold of the Fréedericksz transition in an orthorhombic nematic LC ($G_N = D_{2h}$) holds only when the field $\mathbf{E}$ is directed along the rotational axes C_2 of an orthorhombic nematic LC.

The continuous theory of a nonuniaxial nematic LC was constructed [17] by introducing at each point $P(\mathbf{r})$ of the nematic phase a triplet of the unit vectors $\mathbf{e}^i(\mathbf{r}), i = 1, 2, 3$, connected by six relations imposed on the scalar products $\langle\ ,\ \rangle$

$$\langle \mathbf{e}^i, \mathbf{e}^j \rangle = \zeta_{ij} \ . \tag{1}$$

The expression for the free energy density F_{G_N} of an elastically deformed, nonchiral nematic LC of the point symmetry group G_N is [17]

$$2F_{G_N} = \sum_{i,j,k,l} \Lambda_{ij}^{kl} \langle \mathbf{e}^i, \mathrm{rot}\ \mathbf{e}^j \rangle \langle \mathbf{e}^k, \mathrm{rot}\ \mathbf{e}^l \rangle \ , \tag{2}$$

where Λ_{ij}^{kl} is a fourth-rank tensor, symmetrical with respect to permutation of the upper and lower indices $\Lambda_{ij}^{kl} = \Lambda_{kl}^{ij}$. In the general case (*triclinic* symmetry) this tensor has 45 independent coefficients. This number is reduced for any finite symmetric group G_N, leading to relations between the different components Λ_{ij}^{kl}. These relations can be found [17] for each group G_N, by applying symmetry operations of the G_N to the pseudoscalar products $\langle \mathbf{e}^i, \mathrm{rot}\ \mathbf{e}^j \rangle$. [For the chiral nematic LC, the scalar term $\sum_{i,j} k_{ij} \langle \mathbf{e}^i, \mathrm{rot}\ \mathbf{e}^j \rangle$ should be added to (2). This term also admits a similar symmetry analysis. For the chiral, nonuniaxial nematic LC, such an analysis was done recently in a cumbersome manner [36].]

The orientational part W_w of the energy density of the interaction between nematic LCs with the order parameter Q_w and the external field $\mathbf{E}$ can be expressed as

$$W_w = Q_w\, \varepsilon_w E^w \ , \tag{3}$$

where the dielectric "permittivity" ε_w of the nematic LC appears as an interaction constant and E^w is the wth degree of E. Let us derive the W_w term for every point group G_N of the nonchiral nematic LC more precisely. We will take advantage of the usual parametrization of tensors Q_w by means of M unit vectors $\mathbf{n}^m$ constituting a star of vectors, invariant under the action of the symmetry operations of the corresponding point group G_N. Naturally this star $\mathbf{n}^m$ can coincide with a basic triplet of the unit vectors $\mathbf{e}^i$ as in the case of biaxial and cubic nematic LCs ($M = 3$), but it also can be distinctive as in the case of tetrahedral ($M = 4$) and icosahedral ($M = 15$) nematic LCs. In the last case, of course, there are linear relations in the set of M vectors $\mathbf{n}^m$, providing nevertheless a very convenient expression for W_w to operate with. Using the expressions for Q_2 [2], Q_3 [12], Q_4 [37], and Q_6 [11] we obtain, for the $w = 2$ *nematic* LC of the *biaxial* symmetry group $G_N = D_{2h}$,

$$Q_{2b} = \sum_{m=1}^{M=3} q_m (n_i^m n_j^m - \tfrac{1}{3}\,\delta_{ij}) \ , \ \sum_{m=1}^{M=3} q_m = 0 \ ,$$

$$W_{2b} = \sum_{m=1}^{M=3} \varepsilon_{2m} \langle \mathbf{n}^m, \mathbf{E} \rangle^2 \ , \ \sum_{m=1}^{M=3} \varepsilon_{2m} = 0 \ , \tag{4}$$

and of the *uniaxial* symmetry group $G_N = D_{kd}, D_{(k+1)h}, k \geq 2$,

$$Q_{2u} = q_0\,(n_i n_j - \tfrac{1}{3}\,\delta_{ij}) \ , \ W_{2u} = \varepsilon_{2u} \langle \mathbf{n}, \mathbf{E} \rangle^2 \ ; \tag{5}$$

for the $w = 3$ *tetrahedral nematic* LC $G_N = T_d$,

$$Q_3 = u \sum_{m=1}^{M=4} n_i^m n_j^m n_k^m \ , \ W_3 = \sum_{m=1}^{M=4} \varepsilon_{3m} \langle \mathbf{n}^m, \mathbf{E} \rangle^3 \ ; \tag{6}$$

for the $w = 4$ *cubic nematic* LC $G_N = T_h, O_h$,

$$Q_4 = r \left[\sum_{m=1}^{M=3} n_i^m n_j^m n_k^m n_l^m - \tfrac{1}{5}\,(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \right] ,$$

$$W_4 = \sum_{m=1}^{M=3} \varepsilon_{4m} \langle \mathbf{n}^m, \mathbf{E} \rangle^4 \ ; \tag{7}$$

and for the $w = 6$ *icosahedral nematic* LC $G_N = Y_h$,

$$Q_6 = s \left[\sum_{m=1}^{M=15} n_i^m n_j^m n_k^m n_l^m n_p^m n_t^m - \tfrac{1}{7} \sum \ \delta_{\alpha\beta}\delta_{\gamma\rho}\delta_{\mu\nu} \right]$$

$$W_6 = \sum_{m=1}^{M=15} \varepsilon_{6m} \langle \mathbf{n}^m, \mathbf{E} \rangle^6 \ , \tag{8}$$

where q_0 and q_m, u, r, s are moduli of the tensor order parameters Q_2, Q_3, Q_4, Q_6, respectively. Further, we will accept that all moduli are included as multipliers in the corresponding values of the dielectric permittivity ε_w. In the expression of tensors Q_w we have used M unit vectors $\mathbf{n}^m$. In Q_{2b} these vectors are directed along three rotational axes C_2 of a rectangular parallelepiped, in Q_3 along four rotational axes C_3 of a tetrahedron, in Q_4 along three rotational axes C_2 of a tetrahedron or C_4 of a cube, and in Q_6 along fifteen rotational axes C_2 of an icosahedron. In Q_6 six greek indices $\alpha, \beta, \ldots, \nu$ take all noncoinciding values of six latin indices $i, j, \ldots, t$. This list of order parameters exhausts all possible nonchiral nematic LC phases.

The free energy of a deformed nematic LC of point group G_N in the $\mathbf{E}$ field, calculated per unit area of the surface of the plane-parallel liquid crystal layer, is expressed by the functional

$$J = \frac{1}{2L} \int_{-L}^{L} (F_{G_N} + W_w)\, dz \ , \tag{9}$$

$2L$ being the thickness of nematic layer. The boundary conditions for strong anchoring are

$$\mathbf{e}^i(\pm L) = \mathbf{e}_0^i \ . \tag{10}$$

The standard approach to the problem is to derive appropriate Euler-Lagrange equations for the variational problem with the functional J and the holonomic relationship (1). That leads to a system of nonseparable nonlinear differential equations for the functions $\mathbf{e}^i(z)$. This approach makes it possible to identify the nature of the functions minimizing J and satisfying the conditions (1) and (10), which is equivalent to the application of the Ritz variational method. It leads to an algebraic polynomial of several variables c_i to be investigated by simple analytic methods. That approach was realized for biaxial [25] and tetrahedral [12] nematic LCs. It was also shown that the number of independent variables is equal to 3 and c_i are the amplitudes of the first modes in the Fourier-cosine expansion of the three corresponding Euler angular coordinates $\tau_i(z)$ describing the spatial orientation (Fig. 1) of LC directors $\mathbf{e}^m(z)$,

$$\tau_i(z) = c_i \cos qz + \sum_{k=2}^{\infty} \Phi_k(c_j) \cos^k qz \ , \ i, j = x, y, z \ .$$

Here $\Phi_k(c_j)$ is a homogeneous polynomial of kth order with respect to the variables c_i. It can be determined by solving the corresponding set of an interconnected Euler-Lagrange equation. Finally, it was also noticed that the c_i can be considered as components of the three-dimensional axial vector $\mathbf{c}$. Obviously, the existence of the three-dimensional axial vector that describes the spatial orientation of LC directors can be generalized to the nonchiral nematic LC of the finite arbitrary point symmetry group G_N. Thus the structural (nonthermodynamic) order parameter of nonuniaxial nematic LCs at

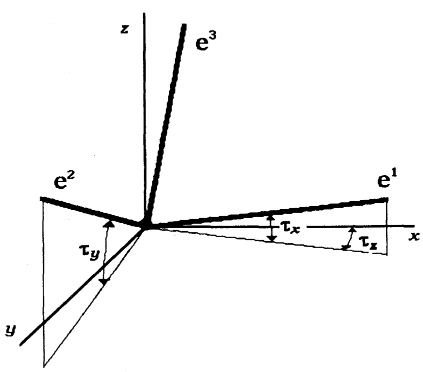

FIG. 1. Perturbed molecular "hedgehog."

the Fréedericksz transition is a three-dimensional axial vector **c**.

For the ordinary uniaxial nematic with a continuous symmetry group $D_{\infty h}$, only two independent variables c_i are essential. The geometry of the Fréedericksz transition in such a nematic LC can be described with one angular variable, e.g., *a planar* nematic LC in *a homeotropic* field.

III. THE CURIE PRINCIPLE AND THE SYMMETRY OF THE FRÉEDERICKSZ TRANSITION

A realization of the approach, discussed in the preceding section, for the nonuniaxial nematic LC with arbitrary point symmetry group G_N, strong anchoring of the LC directors at the boundaries with arbitrary symmetry, and arbitrary orientation of the **E** field is cumbersome from the computational point of view and does not yield any perspective from the theoretical standpoint. The alternative approach is the direct construction of the algebraical polynomial $J(c_i)$ by means of the integrity basis of invariants $I_m(c_i)$ built on the basis of components of the three-dimensional axial vector **c** for different symmetry groups $G_{\rm Fr}$ of the physical system (*symmetry group* of the Fréedericksz transition). Let a polynomial $J(c_i)$ read

$$J = \alpha_i \, c_i \, + \, \beta_{ij} \, c_i c_j + \gamma_{ijk} \, c_i c_j c_k + \delta_{ijkl} \, c_i c_j c_k c_l \, + \cdots . \tag{11}$$

Taking into account the axial nature of the vector **c** one can conclude that, in Jahn's notation [38], the inner symmetry of the tensors $\alpha_i, \beta_{ij}, \gamma_{ijk}, \delta_{ijkl}$, etc., is

$$\alpha_i : \epsilon V; \beta_{ij} : [V^2]; \gamma_{ijk} : \epsilon[V^3]; \delta_{ijkl} : [V^4]; \cdots .$$

Here $[V^n]$ and $\epsilon \, [V^n]$ denote a symmetrical tensor and a pseudotensor of n rank accordingly. A given point symmetry group G_{Fr} leaves, in the above-mentioned classes, a finite number of invariants $I_m(c_i)$ [35].

The symmetry group $G_{\rm Fr}$ must be derived from the symmetry G_{Nb} of nonuniaxial nematic LCs taking into account the boundary conditions and the symmetry G_f of the external field. According to the Curie symmetry principle [35], $G_{\rm Fr}$ is a maximal common subgroup of groups G_{Nb} and G_f,

$$G_{\rm Fr} = \max \{ \, G_{Nb} \bigcap G_f \, \} , \tag{12}$$

and the group G_{Nb} is the maximal common subgroup of G_N and $D_{\infty h}$,

$$G_{Nb} = \max \{ \, G_N \bigcap D_{\infty h} \, \} . \tag{13}$$

Here $D_{\infty h}$ is the symmetry group of two infinite parallel planes. It is easy to show that the elasticity part $J_d = \frac{1}{2L} \int F_{G_N} dz$ of the whole functional J is invariant with respect to the action of the symmetry group G_{Nb}. Thus the group G_{Nb} plays the same role with respect to the J_d as the group G_{Fr} does to the whole functional J.

The rotational axis C_∞ of $D_{\infty h}$ in the three-dimensional space can coincide with the rotational axes C_n of the point group G_N of a nonuniaxial nematic LC. One also can consider the case when this axis C_∞ does not coincide with any rotational axis (which means the coincidence with the trivial axis C_1). One can derive the list of the groups G_{Nb} by use of the relationship scheme of the point groups G_N and their subgroups, considered in [35] and supplemented with the icosahedral group. To concretize the procedure of finding the groups $G_{\rm Fr}$, we begin to deal with intermediate symmetry groups D_{kh}, D_{kd}, as well as with high symmetry groups T_d, T_h, O_h, Y_h contained in the well-known list of finite subgroups of group O(3). Later in this section we will return to the excluded groups D_{kh}, D_{kd} with $k > 6$ to include them in the general scheme. Thus we operate first with rotational axes $C_k, \; k \leq 6$. The complete list of the groups G_{Nb} is presented in Table I, where the Schoenflies notation [35] of point symmetry groups is used.

Now let us turn to the group G_f, which is relatively easy to consider. From the symmetry point of view, there is a difference between the interaction of the **E** field with a tetrahedral nematic LC ($G_N = T_d$) and with the other kinds of LCs from the list (4)–(8) or, in other words, between odd (w=3) and even (w=2,4,6) powers of the field E in the W_w term in (3). The single **E** field in the W_3 term preserves its natural symmetry $G_f = C_{\infty v}$, whereas even w in W_w raises it effectively to $G_f = D_{\infty h}$. There are several ways to intersect the **E** field with LC boundaries, leading to the different groups $G_{\rm Fr}$. Nevertheless, among the 14 point groups G_{Nb} listed in Table I, there are only two, C_1 and C_i, that result in the identity $G_{\rm Fr} = G_{Nb}$ after their intersection with G_f, independently of the direction of the **E** field. For the other 12 groups G_{Nb} this is not true. Three different orientations of the **E** field with respect to the LC boundaries have been distinguished traditionally: *homeotropic*, *planar*, and *oblique*. By means of symmetry rules [35] we obtain the following.

1. Homeotropic orientation of the **E** *field.* For $w = 2, 4, 6$,

$$G_{\rm Fr} = \max \{ \, G_{Nb} \bigcap D_{\infty h} \, \} = G_{Nb} \, , \tag{14}$$

and for $w = 3$,

TABLE I. Symmetry groups $G_{Nb} = \max\{ \, G_N \bigcap D_{\infty h} \, \}$.

ω	G_N	C_1	C_2	C_3	C_4	C_5	C_6
2	D_{2h}	C_i	D_{2h}				
2	D_{3d}	C_i	C_{2h}	D_{3d}			
2	D_{2d}	C_1	D_2		D_{2d}		
2	D_{3h}	C_1	C_{2v}	D_{3h}			
2	D_{4h}	C_i	D_{2h}		D_{4h}		
2	D_{6h}	C_i	D_{2h}				D_{6h}
3	T_d	C_1		C_{3v}	D_{2d}		
4	T_h	C_i	D_{2h}	C_{3i}			
4	O_h	C_i	D_{2h}	D_{3d}	D_{4h}		
6	Y_h	C_i	D_{2h}	D_{3d}		D_{5d}	

$$G_{\rm Fr} = \max \{ G_{Nb} \bigcap C_{\infty v} \} = \begin{cases} C_{2v} & \text{if } G_{Nb} = D_{2d} \\ C_{3v} & \text{if } G_{Nb} = C_{3v}. \end{cases}$$

[For nonchiral tetrahedral nematic LCs ($w = 3$) the symmetry groups $G_{\rm Fr}$ of the Fréedericksz transition were obtained in [12].]

2. Planar orientation of the **E** *field.* The middle symmetry of the main part of point groups G_{Nb} makes it necessary to divide this case into two: (a) the **E** field is directed along a rotational axis C_2 of an unperturbed LC or is situated in its symmetry plane (e.g., for C_{2h}) and (b) the **E** field does not coincide with any symmetry element of the group G_{Nb}. For $w = 2$, in accordance with cases (a) and (b) the groups $G_{Nb} = D_{2d}, D_{3d}, D_{3h}, D_{4h}, D_{6h}$ give rise to the relations

$$\begin{aligned} D_{2d}, D_{3h}&: G_{\rm Fr(a)} = G_{\rm Fr(b)} = D_2; \\ D_{3d}, D_{4h}, D_{6h}&: G_{\rm Fr(a)} = G_{\rm Fr(b)} = D_{2h} \end{aligned} \tag{15}$$

due to azimuthal degeneration of the W_2 term with respect to the direction of the **E** field in the plane. The group C_{2h} also gives rise to the identity $G_{\rm Fr} = C_{2h}$ due to the relation $C_{2h} \subset D_{\infty h}$, when the rotational axes C_∞ of both these groups are mutually orthogonal. The remaining three groups C_{2v}, D_2, D_{2h} lead to the following groups $G_{\rm Fr(a)}, G_{\rm Fr(b)}$:

$$\begin{aligned} C_{2v}&: G_{\rm Fr(a)} = C_{2v}, G_{\rm Fr(b)} = C_2; \\ D_2&: G_{\rm Fr(a)} = D_2, G_{\rm Fr(b)} = C_2; \\ D_{2h}&: G_{\rm Fr(a)} = D_{2h}, G_{\rm Fr(b)} = C_{2h} . \end{aligned} \tag{16}$$

For $w = 3$,

$$\begin{aligned} C_{3v}&: G_{\rm Fr(a)} = G_{\rm Fr(b)} = C_1; \\ D_{2d}&: G_{\rm Fr(a)} = D_2, G_{\rm Fr(b)} = C_2 . \end{aligned} \tag{17}$$

For $w = 4$,

$$\begin{aligned} D_{2h}&: G_{\rm Fr(a)} = D_{2h}, G_{\rm Fr(b)} = C_{2h}; \\ C_{3i}&: G_{\rm Fr(a)} = G_{\rm Fr(b)} = C_i; \\ D_{4h}&: G_{\rm Fr(a)} = D_{2h}, G_{\rm Fr(b)} = C_{2h}; \\ D_{3d}&: G_{\rm Fr(a)} = C_{2h}, G_{\rm Fr(b)} = C_i. \end{aligned} \tag{18}$$

For $w = 6$,

$$\begin{aligned} D_{2h}&: G_{\rm Fr(a)} = D_{2h}, G_{\rm Fr(b)} = C_{2h}; \\ D_{3d}&: G_{\rm Fr(a)} = C_{2h}, G_{\rm Fr(b)} = C_i; \\ D_{5d}&: G_{\rm Fr(a)} = C_{2h}, G_{\rm Fr(b)} = C_i. \end{aligned} \tag{19}$$

3. Oblique orientation of the **E** *field.* For $w = 2, 4, 6$,

$$G_{\rm Fr} = \max \{ G_{Nb} \bigcap C_i \} = \begin{cases} C_i & \text{if } C_i \subset G_{Nb} \\ C_1 & \text{otherwise}, \end{cases}$$

and for $w = 3$,

$$G_{\rm Fr} = C_1 . \tag{20}$$

Thus we have only 15 symmetry groups $G_{\rm Fr}$ of the Fréedericksz transition: 14 of them enter into the list of G_{Nb} and one (C_2) appears in (17) and (16). The list of the external field orientations leads to the most wide variety of symmetry groups $G_{\rm Fr}$ of the Fréedericksz transition only when the field **E** is oriented homeotropically to the LC boundaries; then the $G_{\rm Fr}$ and G_{Nb} groups coincide. In the other cases, the groups $G_{\rm Fr}$ are only a part of the presented list (Table I) and also can include the monoclinic group C_2.

Return now to the uniaxial nematic LC phases ($w = 2$) of the point groups $G_N = D_{kh}, D_{kd}$ with arbitrary k. It is well known from a crystallographic point of view that it is more convenient to operate with even and odd k separately. Computing the groups G_{Nb} in accordance with (12), one easily obtains, for $C_\infty \parallel C_1$,

$$\begin{aligned} G_{Nb}(D_{2nh}) &= G_{Nb}(D_{(2n+1)d}) = C_i, \\ G_{Nb}(D_{2nd}) &= G_{Nb}(D_{(2n+1)h}) = C_1 , \end{aligned}$$

and for $C_\infty \parallel C_2$,

$$\begin{aligned} G_{Nb}(D_{2nh}) &= D_{2h}, G_{Nb}(D_{2nd}) = D_2, \\ G_{Nb}(D_{(2n+1)h}) &= C_{2v}, G_{Nb}(D_{(2n+1)d}) = C_{2h} . \end{aligned}$$

These groups were already listed in Table I. The new groups G_{Nb} appear as a result of operating with the rotational axis C_∞ of the group $D_{\infty h}$, directed along the rotational axes C_{2n} or C_{2n+1} of the considered nematic LC phases. In this case we really obtain $G_{Nb}(G_N) = G_N$. Now the last step is to find the corresponding groups $G_{\rm Fr}$. It is easy to show that for the homeotropic orientation of the **E** field these groups give rise to the identity $G_{\rm Fr} = G_{Nb}$; the planar orientation of the field causes relations similar to (15),

$$\begin{aligned} D_{2kd}, D_{(2k+1)h}&: G_{\rm Fr(a)} = G_{\rm Fr(b)} = D_2; \\ D_{(2k+1)d}, D_{2kh}&: G_{\rm Fr(a)} = G_{\rm Fr(b)} = D_{2h}; \end{aligned}$$

and an oblique orientation of the field leads to the relationship (20) for $w = 2$.

TABLE II. Symmetry groups $G_{\rm Fr}$ for ordinary uniaxial nematic $G_N = D_{\infty h}$. **h** is an orthogonal vector to the LC boundary plane and [**h**, **e**, **E**] is a mixed product of the three vectors **h**, **e**, and **E**.

Initial orientation	**e** homeotropic	**e** planar	**e** planar	**e** oblique	**e** oblique
E homeotropic	$D_{\infty h}$	D_{2h}	D_{2h}	C_{2h}	C_{2h}
E planar	D_{2h}	$\langle \mathbf{e}, \mathbf{E} \rangle = 0$ D_{2h}	$\langle \mathbf{e}, \mathbf{E} \rangle \neq 0$ C_{2h}	$[\mathbf{h}, \mathbf{e}, \mathbf{E}] = 0$ C_{2h}	$[\mathbf{h}, \mathbf{e}, \mathbf{E}] \neq 0$ C_i
E oblique	C_{2h}	$[\mathbf{h}, \mathbf{e}, \mathbf{E}] = 0$ C_{2h}	$[\mathbf{h}, \mathbf{e}, \mathbf{E}] \neq 0$ C_i	$[\mathbf{h}, \mathbf{e}, \mathbf{E}] = 0$ C_{2h}	$[\mathbf{h}, \mathbf{e}, \mathbf{E}] \neq 0$ C_i

We complete this section by including in the present scheme the Fréedericksz transition in the ordinary uniaxial nematic with symmetry group $G_N = D_{\infty h}$. This case belongs to the interaction degree $w = 2$. In Table II we present the symmetry groups $G_{\rm Fr}$ of the Fréedericksz transition in the uniaxial nematic LC for three possible orientations—homeotropic, planar, and oblique—of the **e** director and the **E** field with respect to the LC boundaries. Thus, in this case, we have only four different symmetry groups $G_{\rm Fr}$ of the Fréedericksz transition: $D_{\infty h}, D_{2h}, C_{2h}, C_i$.

IV. STATIONARY STATES AND STRUCTURAL TRANSITIONS

Let us construct a free energy polynomial J of a deformed nonchiral nematic LC subjected to an **E** field of a specified orientation with respect to the LC boundaries. For this purpose let us use the integrity basis of the invariants $I_m(c_i)$ [35,39], built on the basis of a three-dimensional axial vector **c** and invariant with respect to the action of the symmetry groups $G_{\rm Fr}$, where $c_\rho \cos\phi = c_x$, and $c_\rho \sin\phi = c_y$. According to the trigonometric identities [40], we can represent

$$c_\rho^k \cos k\phi = \sum_{m=0}^{2m\leq k} (-1)^m \, C_k^{2m} \, c_y^{2m} \, c_x^{k-2m},$$

$$c_\rho^k \sin k\phi = \sum_{m=0}^{2m\leq k-1} (-1)^m \, C_k^{2m+1} \, c_y^{2m+1} \, c_x^{k-2m-1},$$

where C_k^m are the binomial coefficients. The limiting case of the ordinary uniaxial nematic LC $G_N = D_{\infty h}$, which follows from D_{kh} by taking the limit $k \to \infty$, leads to only two independent invariants of second degree

$$c_z^2, \;\; c_x^2 \; + \; c_y^2 \; . \tag{21}$$

According to the Landau theory of phase transitions, the main contribution of the W_w term to the polynomial $J(c_i)$ is due to the sign alternating coefficients of the quadratic invariants $(q^2K - \varepsilon_w E^w)|c|^2$, where $|c|^2 = c_i^2$ for the one-component order parameter and $|c|^2 = c_\rho^2 = c_x^2 + c_y^2$ for the two-component order parameter, K is a linear combination of the elasticity modules Λ_{ij}^{kl}, and $q = \frac{\pi}{2L}$ is the wave number. Let us illustrate the last statement by some examples. The term quadratic in c_i for the biaxial nematic LC $G_N = G_{Nb} = G_{\rm Fr} = D_{2h}$ reads [25]

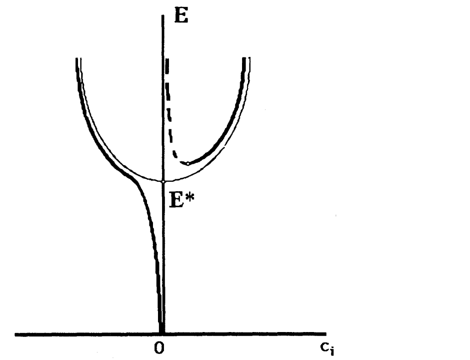

FIG. 2. Bifurcation tree of the Fréedericksz transition in a nematic LC with low symmetry.

$$b_1 c_x^2 \; + \; b_2 c_y^2 \; + \; b_3 c_z^2 \; , \tag{22}$$

where

$$b_1 = q^2(K_1 + K_{23}) - \varepsilon_{2,1}E^2 \; ,$$
$$b_2 = q^2(K_2 + K_{13}) - \varepsilon_{2,2}E^2 \; ,$$
$$b_3 = q^2(K_{11} + K_{22}) - \varepsilon_{2,3}E^2 \; .$$

K_i, K_{ij}, and $\varepsilon_{2,i}$ are, respectively, the elasticity modules and dielectric permittivities of biaxial nematic LCs. The term quadratic in c_i for the tetrahedral nematic LCs $G_N = T_d$ and $G_{Nb} = G_{\rm Fr} = C_{3v}$ reads [12]

$$t_1(c_x^2 \; + \; c_y^2) \; + \; t_2 c_z^2 \; , \tag{23}$$

where

$$t_1 = q^2(3K_1 + K_2 + 2K_3) - \varepsilon_3 E^3 \; , \; t_2 = 2q^2(2K_2 + K_3) \; .$$

K_i and ε_3 are the elasticity modules and dielectric permittivities of tetrahedral nematic LCs.

Further study of the polynomial $J(c_i)$ allows us to answer the following questions: (i) Do the bifurcations (structural transitions) exist at the Fréedericksz transition described by the symmetry group $G_{\rm Fr}$? (ii) If the bifurcations exist, what kind (primary or secondary) are they? (iii) What orders of structural transitions occurring at the bifurcation points exist? The main features of the polynomials $J(c_i)$, invariant with respect to the certain symmetry group $G_{\rm Fr}$, are well known [26] from the

TABLE III. Invariant polynomials.

Symmetry group $G_{\rm Fr}$	Integrity basis of invariants $I_m(c_i)$
C_1, C_i	c_z, c_x, c_y
C_2, C_{2h}	$c_z, c_x^2, c_y^2, c_x c_y$
C_{2v}, D_2, D_{2h}	$c_z^2, c_x^2, c_y^2, c_z c_x c_y$
C_{3i}	$c_z, c_\rho^2, c_\rho^3 \cos 3\phi, c_\rho^3 \sin 3\phi$
D_{2kd}, D_{4kh}	$c_z^2, c_\rho^2, c_\rho^{4k} \cos 4k\phi, c_z c_\rho^{4k} \sin 4k\phi$
$C_{(2k+1)v}, D_{(2k+1)d}$	$c_z^2, c_\rho^2, c_\rho^{2k+1} \cos(2k+1)\phi, c_z c_\rho^{2k+1} \sin(2k+1)\phi$
$D_{(2k+1)h}, D_{2(2k+1)h}$	$c_z^2, c_\rho^2, c_\rho^{2(2k+1)} \cos 2(2k+1)\phi, c_z c_\rho^{2(2k+1)} \sin 2(2k+1)\phi$

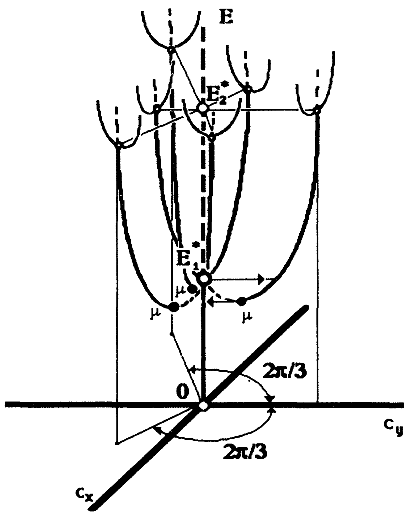

FIG. 3. Bifurcation tree of the Fréedericksz transition with symmetry group $G_{\mathrm{Fr}} = C_{3v}$.

theory of the phase transition in solid states. Therefore, we omit intermediate calculations and present only the main features of these transitions.

The presence of a linear invariant c_z in $I_m(c_i)$ for point groups $C_1, C_i, C_2, C_{2h}, C_{3i}$ leads to the continuous nonthreshold Fréedericksz transition. In other words, the disturbance of homogeneous LC structure occurs in any weak **E** field. As shown in [28], one can expect in such a system only finite jumps in the order parameter between different bifurcation branches (Fig. 2).

The presence of the cubic invariant c_x^3 in $I_m(c_i)$ for point groups C_{3v}, D_{3d} leads to the strong first-order transition. Besides, in such a system a secondary bifurcation is permitted as a first- or second-order transition (Fig. 3). It must be noted that the Fréedericksz transition in rhombohedral nematics of the considered groups can occur as a second-order transition at the manifold of a small dimension, defined on the multiparametrical space of the problem. This kind of transition is similar to the thermo-

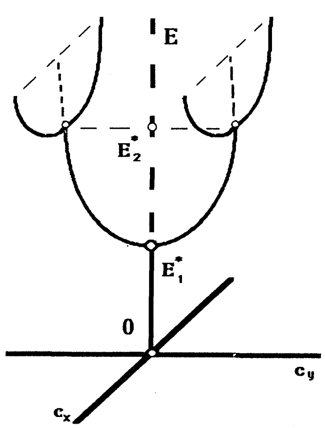

FIG. 4. Bifurcation tree of the Fréedericksz transition with symmetry group $G_{\mathrm{Fr}} = C_{2v}$, $w = 2$.

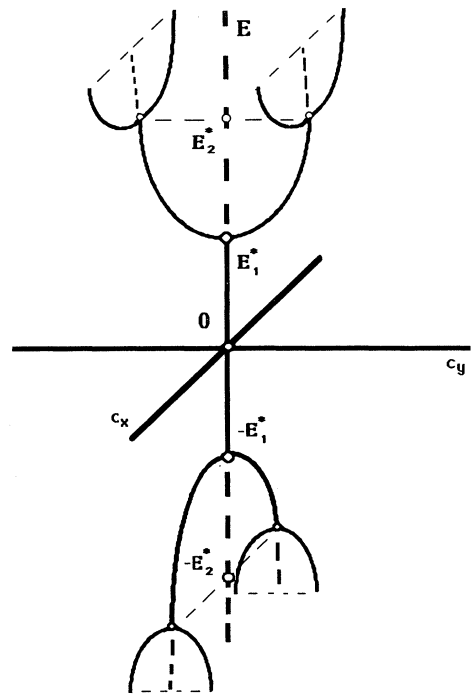

FIG. 5. Bifurcation tree of the Fréedericksz transition with symmetry group $G_{\mathrm{Fr}} = C_{2v}$, $w = 3$.

dynamical second-order phase transition from isotropic liquid into the biaxial nematic at the tricritical point, where isotropic liquid, biaxial nematic LCs, and two uniaxial nematic LCs (calamitic and discotic) coexist with a very narrow stability region of biaxial nematic in the vicinity of the transition [2].

The presence of the nonlinear invariant $c_x c_y c_z$ in $I_m(c_i)$ for point groups C_{2v}, D_2, D_{2h} allows, nevertheless, in the vicinity of the primary bifurcation point the second-order transition [25], provided coefficients at the invariants of fourth degree, which enter the polynomial $J(c_i)$, are of the correct sign. In the vicinity of the secondary bifuracation points the Fréedericksz transition can evolve either as a first- or second-order transition (Fig. 4). The case of the group C_{2v} with the cubic term E^3 of the **E**-field interaction with a tetrahedral nematic LC, which leads to the specific bifurcation tree (Fig. 5), has to be described separately. For the intermediate symmetry groups $G_{\mathrm{Fr}} = D_{2d}, D_{(k+1)d}, D_{kh}$ $(k \geq 3)$ the nonlinear invariant c_x^{k+1} does not forbid the existence of second-order transitions (Fig. 6) at the primary and secondary bifurcation points.

Since the rotation around the axis C_∞ is not relevant for the continuous symmetry group $G_{\mathrm{Fr}} = D_{\infty h}$, the variable c_z does not enter into $J(c_i)$. There is actually only one quadratic invariant c_ρ^2 describing deviation of the director **e** from the direction of the **E** field. Thus, in this case there is only one bifurcation point [28], where the Fréedericksz transition is of second or (taking into account an electrical conductivity [41]) first order (Fig. 7).

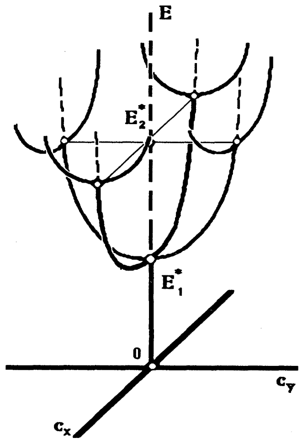

FIG. 6. Bifurcation tree of the Fréedericksz transition with symmetry group $G_{\rm Fr} = D_{4h}$.

V. THERMAL FLUCTUATIONS OF THE NONUNIAXIAL NEMATIC DIRECTORS

Let us consider the disturbances of the Fréedericksz transition due to the thermal fluctuations of nonchiral nematic LC directors in the x-y plane. For this purpose we take into account the two-dimensional gradients $\nabla_{x,y}\mathbf{c}$ of the order parameter. Preserving the rigidity conditions (1) of the nematic LC directors and the boundary conditions of strong anchoring (10), we obtain an expression for a total free energy $\mathcal{J}$ of a deformed, infinite plane-parallel, nematic LC layer subjected to an external field

$$\mathcal{J} = \int\!\!\int_S [J(c_i) + j(\nabla_{x,y}c_i)]\, dx\, dy \;, \tag{24}$$

where S is a surface area of the LC layer ($S \gg L^2$) and $c_i = c_i(x,y)$, $i = x,y,z$. Naturally, the invariants $I_m(c_i)$, which were counted in Table III, also depend on the surface coordinates x, y. In order to study the features of the behavior of the vector order parameter $\mathbf{c}$ in the vicinity of the transition into the low symmetrical phase, it is sufficient to use for the angular coordinates τ_i a one-mode approximation in z,

$$\tau_i(x,y,z) \propto c_i(x,y)\cos qz \;, \quad i = x,y,z \;, \tag{25}$$

and to restrict ourselves to the quadratic approximation in c_i and $\nabla_{x,y}\mathbf{c}$.

As in Sec. IV, let us use the integrity basis of the invariants, built on the basis of the two-dimensional gradients $\nabla_{x,y}\mathbf{c}$ and invariant with respect to action of the symmetry groups $G_{\rm Fr}$. Since the influence of the fluctuations is critical in the vicinity of the second-order transition (*Ginzburg criterion*) and negligible in the vicinity of the strong first-order transition or the continuous nonthreshold transition, we will pay attention here to the biaxial, tetragonal, and hexagonal symmetry groups $G_{\rm Fr}$ discussed in Sec. IV. These point symmetry groups admit the general expression for the gradient part $j(\nabla_{x,y}c_i)$ in the functional $\mathcal{J}$ as follows:

$$\begin{aligned} j = {} & a_1(\partial_x c_x)^2 + a_2(\partial_y c_y)^2 + a_3(\partial_x c_x)(\partial_y c_y) \\ & + a_4(\partial_y c_x)^2 + a_5(\partial_x c_y)^2 + a_6(\partial_y c_x)(\partial_x c_y) \\ & + a_7(\partial_x c_z)^2 + a_8(\partial_y c_z)^2 \;, \end{aligned} \tag{26}$$

where the coefficients a_i are independent for the point groups C_{2v}, D_{2h} and are related for the intermediate symmetry groups D_{2d}, D_{4h},

$$a_1 = a_2 \;, \quad a_4 = a_5 \;, \quad a_7 = a_8 \;, \tag{27}$$

and D_{3h}, D_{6h},

$$a_1 = a_2, \;\; a_4 = a_5, \;\; a_7 = a_8, \;\; a_1 = a_3 + a_4 + a_6 \;. \tag{28}$$

The coefficients a_i are the linear combinations of the elasticity modules of nonuniaxial nematic LCs. As can be shown for the biaxial nematic LCs with the free energy density $F_{D_{2h}}$, derived in [17],

$$2F_{D_{2h}} = \sum_{i=1}^{3} K_i \,{\rm div}^2\, \mathbf{e}^i + \sum_{i,j=1}^{3} K_{ij}\, \langle \mathbf{e}^i, {\rm rot}\, \mathbf{e}^j\rangle^2 \;, \tag{29}$$

the coefficients a_i read

$$\begin{aligned} & a_1 = K_3 + K_{21}, \;\; a_2 = K_3 + K_{12}, \;\; a_3 = 2K_3, \\ & a_4 = K_{33} + K_{11}, a_5 = K_{33} + K_{22}, \;\; a_6 = -2K_{33}, \\ & a_7 = K_2 + K_{31}, \;\; a_8 = K_1 + K_{32} \;. \end{aligned} \tag{30}$$

In order to reduce the functional $\mathcal{J}$ to the canonical form let us perform the Fourier transform

$$c_j(x,y) = \frac{1}{\sqrt{S}} \sum_{\mathbf{k}} \eta_{j,\mathbf{k}} \exp\left(i\,\langle \mathbf{k},\rho\rangle\right) \;, \;\; j = x,y,z \;, \tag{31}$$

where $\langle \mathbf{k},\rho\rangle = k_x x + k_y y$, ρ and $\mathbf{k}$ are the vectors in real and reciprocal two-dimensional space, respectively.

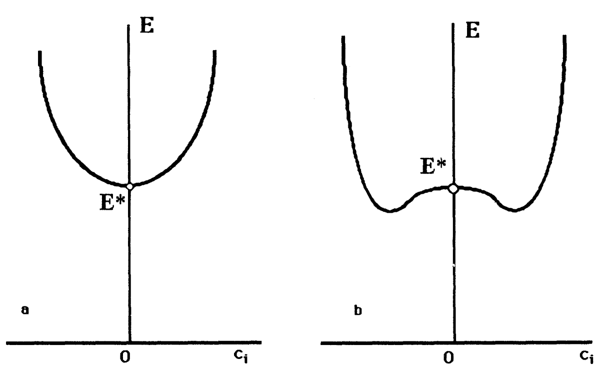

FIG. 7. Bifurcation tree of the Fréedericksz transition in a uniaxial nematic LC: (a) second order; (b) first order.

Inserting now the last transform into the expression for $\mathcal{J}$ and integrating with the substitution of the integral representation of the δ function

$$\delta(\mathbf{k}-\mathbf{k}') = \frac{1}{S}\iint_S \exp\left[\, i\,\langle(\mathbf{k}-\mathbf{k}'),\rho\rangle\right] d^2\rho\ , \qquad (32)$$

we find a quadratic form in six-dimensional space $(\eta_{j,\mathbf{k}},\ \eta^*_{j,\mathbf{k}})$

$$\mathcal{J} = \frac{1}{2}\sum_{\mathbf{k}}(g_{xx}|\eta_{x,\mathbf{k}}|^2 + g_{xy}\eta_{x,\mathbf{k}}\eta^*_{y,\mathbf{k}} + g_{yy}|\eta_{y,\mathbf{k}}|^2 + g_{yx}\eta_{y,\mathbf{k}}\eta^*_{x,\mathbf{k}} + g_{zz}|\eta_{z,\mathbf{k}}|^2)\ , \qquad (33)$$

where

$$\begin{aligned} g_{xx} &= a_1k_x^2 + a_4k_y^2 + b_1,\ \ g_{yy} = a_5k_x^2 + a_2k_y^2 + b_2\ ,\\ g_{xy} &= a_3k_xk_y,\ \ g_{yx} = a_6k_xk_y,\\ g_{zz} &= a_7k_x^2 + a_8k_y^2 + b_3\ , \end{aligned} \qquad (34)$$

and the coefficients b_i were defined in (22). Diagonalizing the quadratic form for $\mathcal{J}$ we obtain in the new coordinate basis $(\nu_{j,\mathbf{k}},\ \nu^*_{j,\mathbf{k}})$

$$\mathcal{J} = \frac{1}{2}\sum_{\mathbf{k}}\left(\lambda_x|\nu_{x,\mathbf{k}}|^2 + \lambda_y|\nu_{y,\mathbf{k}}|^2 + \lambda_z|\nu_{z,\mathbf{k}}|^2\right)\ , \qquad (35)$$

where

$$\begin{aligned} \lambda_{x,y}^2 &= g_{xx}^2 + g_{yy}^2 + g_{xy}^2 + g_{yx}^2\\ &\pm[(g_{xx}-g_{yy})^2 + (g_{xy}+g_{yx})^2]^{\frac{1}{2}}\\ &\times[(g_{xx}+g_{yy})^2 + (g_{xy}-g_{yx})^2]^{\frac{1}{2}}\ ,\end{aligned}$$

$$\lambda_z = g_{zz}\ . \qquad (36)$$

Now it is easy to show that $\lambda_x^2 > 0,\ \lambda_y^2 > 0$. This follows from the definition of $\lambda_{x,y}$ and also from the identity $\lambda_x^2\ \lambda_y^2 = 4\ (g_{xy}g_{yx} - g_{xx}g_{yy})^2$, which can be easily checked. The equipartition theorem gives the mean-square fluctuation $\langle|\nu_{j,\mathbf{k}}|^2\rangle$

$$\langle|\nu_{j,\mathbf{k}}|^2\rangle = \frac{\sqrt{2}\,T}{\lambda_j}\ ,\ \ j = x,y,z\ . \qquad (37)$$

Taking into account the inequalities $g_{xx} \neq g_{yy}$ and $g_{xy} \neq g_{yx}$, following from the definition of g_{ij}, we can conclude $\lambda_x \neq \lambda_y$ even in the case of the intermediate symmetry groups $G_{\rm Fr}$. The last statement means that the Fréedericksz transition occurs as a second-order transition that evolves via one of three one-dimensional irreducible representations of the groups mentioned above. Thus, taking into account the thermal fluctuations of the LC directors in the Fréedericksz transition with symmetry groups $G_{\rm Fr} = D_{2d}, D_{(k+1)d}, D_{kh},\ k \geq 3$ this leads to the removal of the degeneracy of the order parameter $\mathbf{c}$ in the x-y plane, which is orthogonal to the mean symmetry axis. It is a consequence of the elasticity anisotropy of the nonuniaxial nematic LC, where the elasticity modules Λ_{ij}^{kl} arrange a fourth-rank tensor.

It follows from (37) that the fluctuations of all three modes ν_j behave in a critical manner in the vicinity of the corresponding thresholds $E_{{\rm th},j}$, which are determined by the equations $b_j = 0$, where b_j are determined in (22). In contrast to the scalar order parameter, here we have in $\mathbf{r}$ space three correlation functions $\Gamma_j(\mathbf{r}_1,\mathbf{r}_2)$ with their own correlation lengths ξ_j. For the longitudinal mode ν_z the correlation length has a standard form $\xi_z = \frac{L}{\pi}\,(1 - E^2/E^2_{{\rm th},z})^{-0.5}$. For two transverse modes $\nu_{x,y}$ the expression for $\xi_{x,y}$ cannot be found analytically.

VI. CONCLUSION

On the basis of the phenomenological theory of nematic LCs, the theory of the Fréedericksz transition in nonchiral nematic LCs for all possible symmetries of LC point groups G_N, with strong boundary LC orientation at the infinite plane-parallel plates of slab and arbitrary direction of an applied $\mathbf{E}$ field, has developed. The advantage of the integrity basis of the invariant polynomial method for the symmetry group $G_{\rm Fr}$ of the Fréedericksz transition, which is built by taking into account the symmetry of the strong boundary conditions of LCs as well as the symmetry of the external $\mathbf{E}$ field, has been employed. The number and the order of the bifurcations in the Fréedericksz transition have been derived on the basis of the simple group-theoretical considerations.

The thermal fluctuations of nonchiral nematic LC directors in the vicinity of the second-order Fréedericksz transition, with intermediate symmetry groups $G_{\rm Fr} = D_{2d}, D_{(k+1)d}, D_{kh},\ k \geq 3$, lead to the removal of the degeneracy of the order parameter in the plane orthogonal to the mean symmetry axis. It is a consequence of the elasticity anisotropy of the nonuniaxial nematic LC.

From the experimental standpoint the study of the Fréedericksz transition in nonchiral nematic LCs provides another sensitive tool for the identification of the inner symmetry G_N of LCs. The sequence of the structural transitions (the bifurcation points on the bifurcation tree) leads to a decrease of the external symmetry $G_{\rm Fr}$ of the LC cell. This is reflected in the change of its optical properties, e.g., a birefringence. Such experiments were carried out in blue cubic phases of LCs [24]: the orientation of a cubic LC predicted in [22] is precisely the same as that observed in [24]. Unfortunately, the author does not know other experiments where the Fréedericksz transition was observed in nonuniaxial nematic LCs. On the one hand, this is connected with rather specific thermodynamic conditions that had to be imposed on the LC, e.g., a very narrow temperature gap where biaxial [3,7] and cubic [9] LCs exist. On the other hand, it reflects a gradual decrease of pragmatic interest to this branch (LCs) of condensed matter in the past decade. Nevertheless, it is worth noting that during the past decade the question about the inner symmetry G_N of the quasicrystalline structure of the foggy blue phase, where G_N is expected to be icosahedral [10], has remained unsolved. It seems now that this foggy blue phase could be a good candidate for examination of its inner symmetry via the Fréedericksz transition. Indeed, as shown in Table I, the icosahedral symmetry gives rise to four different point groups G_{Nb} in accordance with the direction of the applied electric field along four different rotational axes of possible order: C_1, C_2, C_3, and C_5. These four groups $G_{Nb} = C_i, D_{2h}, D_{3d}, D_{5d}$ correspond to four different bifurcation trees Figs. 2, 4, 3, and 6, respectively. Thus the experimental finding of four different kinds of Fréedericksz transition in the foggy blue phase, which are similar to those mentioned above, could be good evidence for the existence of the icosahedral symmetry.

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